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LETTER TO THE EDITOR

# Symmetry breaking patterns for third-rank totally antisymmetric tensor representations of unitary groups

C J Cummins and R C King
University of Southampton, Southampton, SO9 5NH, England

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**Abstract.** Some possible symmetry breaking patterns for unitary group gauge theoretic models based on Higgs scalars in the third-rank totally antisymmetric tensor representations of $U(n)$ are studied. The critical points are expressed in terms of a single parameter $\xi$ such that $1/n \leqslant \xi \leqslant 1/3$. It is shown that spontaneous symmetry breaking may take place from $SU(n)$ to $SU(3)\times SU(n-3)$ for all $n$, from $SU(6)$ to $SU(3)\times SU(3)$, from $SU(7)$ to $G_2$, from $SU(8)$ to $SU(3)$ and from $SU(9)$ to $U(1)\times U(1)\times U(1)\times U(1)$. The absolute minima are shown to be degenerate for $n \geqslant 9$, and remarks are made concerning a conjecture of Michel.

In building unified gauge theoretic models of the fundamental interactions the breaking of the symmetry with respect to the original gauge group $G$ is often achieved through the introduction of a set of Higgs scalar fields $\varphi$ which form the basis of some representation $\lambda_G$ of $G$. There then exists a corresponding $G$-invariant potential $V(\varphi)$ whose absolute minimum occurs at some non-zero value of $\varphi$. The fields are said to acquire this non-zero vacuum expectation value and the symmetry breaks spontaneously from $G$ to the little group, or isotropy group, $H_\varphi$ of $\varphi$, at the absolute minimum. The requirement of renormalisability is such that $V(\varphi)$ is necessarily a multinomial in the components of $\varphi$ of degree no greater than four.

It is of some mathematical interest to explore systematically the possible symmetry breaking schemes for a variety of gauge groups $G$ and representations $\lambda_G$. A start was made on this programme by Li (1974) who in considering the groups $SU(n)$ and $SO(n)$ examined Higgs fields in the defining, adjoint and the symmetric and antisymmetric second-rank tensor representations. Proceeding to the $k$th rank antisymmetric tensor representations of these groups might seem to be a very modest extension of these results. However, even the first step of analysing the third-rank antisymmetric tensor representations is non-trivial as evidenced by the preliminary results of Kim (1982a) and Jetzer *et al* (1983). A major difficulty is that, despite the claim of Kim, there does not appear to exist a canonical form for a third-rank totally antisymmetric tensor of $SU(n)$ analogous to that which exists in the second-rank case. However, some such forms are available (Schouten 1931, Gurevich 1934, 1935, 1964) for the tensors of $GL(n)$ in the cases $n \leqslant 8$. These forms are used here to obtain critical points of $V(\varphi)$.

We also show that it is possible to obtain bounds on the absolute minima of $V(\varphi)$ and then to saturate these bounds, thereby establishing possible symmetry breaking patterns. This is done by considering the critical points already obtained for $n \leqslant 8$ and by an *ad hoc* procedure for $n \geqslant 9$. In this way we demonstrate a rich variety of symmetry breaking patterns.

Our results have a bearing on a conjecture of Michel (1979). This states that if $\varphi$ forms a basis of a representation of the group $G$ that is irreducible over the reals, and if $V(\varphi)$ is a real fourth-degree $G$-invariant potential that is bounded from below with a maximum at the origin, then the little group of the absolute minimum of $V(\varphi)$ is a maximal little group of the full symmetry group $\tilde{G}$ of $V(\varphi)$. We note that in other formulations of this conjecture (Slansky 1981) the requirement that $\tilde{G}$ rather than $G$ is the relevant group is replaced by the requirement that radiative corrections be considered. In general these two requirements are not equivalent. We show that there is the possibility of breaking from $\mathrm{U}(n)$ to a little group whose Lie algebra is non-maximal amongst the set of all Lie algebras of little groups (we shall refer to such Lie algebras as stationary subalgebras). Hence the little group itself can only be maximal by virtue of finite group elements not connected to the identity. This would seem to limit the usefulness of Michel's conjecture.

The most general $\mathrm{SU}(n)$ invariant potential of fourth degree in the components $\varphi_{ijk}$ of a totally antisymmetric third-rank tensor takes the form

$$V(\varphi)=-\tfrac{1}{2}\mu^2\sum_{ijk}\varphi_{ijk}\varphi^{ijk}+\tfrac{1}{4}\lambda_1\left(\sum_{ijk}\varphi_{ijk}\varphi^{ijk}\right)^2+\tfrac{1}{4}\lambda_2\sum_{\substack{ijk\\pql}}\varphi_{ijk}\varphi^{ijl}\varphi_{pql}\varphi^{pqk} \tag{1}$$

with all the indices summed over the values $1, 2, \ldots, n$ and necessarily $n \geqslant 4$.

If the potential is to be bounded from below, then $\lambda_2 > 0$ requires $n\lambda_1+\lambda_2>0$ and $\lambda_2<0$ requires $3\lambda_1+\lambda_2>0$. For the potential to have a local maximum at the origin it is required that $\mu^2>0$. The components $\varphi_{ijk}$ form the basis of the $\{1^3\}$ representation of $\mathrm{SU}(n)$ of dimension $N=n(n-1)(n-2)/6$, whilst the components $\varphi^{ijk}$ $(=\varphi_{ijk}{}^*)$ form a basis of the contragredient representation $\{\bar{1}^3\}$, which in $\mathrm{SU}(n)$ is equivalent to $\{1^{n-3}\}$.

The complete representation $\{1^3\}+\{\bar{1}^3\}$ is irreducible over the reals and has dimension $2N$. The complete set of orthogonal transformations of the corresponding real vector space constitutes the group $\mathrm{O}(2N)$. The relevant group–subgroup chain takes the form $\mathrm{O}(2N)\supset \mathrm{SO}(2N)\supset \mathrm{U}(N)\supset \mathrm{U}(n)\supset \mathrm{SU}(n)$, with the corresponding branchings

$$[1]\to[1]\to\{1\}+\{\bar{1}\}\to\{1^3\}+\{\bar{1}^3\}\to\{1^3\}+\{1^{n-3}\}.$$

Each link of this chain involves a maximal embedding so that the full symmetry group $\tilde{G}$ of the potential (1), in the case where the original symmetry group $G$ is $\mathrm{SU}(n)$, can only be one of the listed groups (provided we only admit orthogonal representations). If $\lambda_2\neq 0$ then we rule out $\mathrm{O}(2N)$, $\mathrm{SO}(2N)$, and $\mathrm{U}(N)$ since they each possess only a single fourth-degree invariant, namely the term in $\lambda_1$. In this case it is then clear that $\mathrm{U}(n)$ is the full symmetry group $\tilde{G}$ of $V(\varphi)$.

We next define the parameter $\xi$, and derive the bounds on the absolute minima of $V(\varphi)$. The potential (1) can be written in the form

$$V(\varphi)=\tfrac{1}{2}\mu^2\sum_p\theta^p_p+\tfrac{1}{4}\lambda_1\left(\sum_p\theta^p_p\right)^2+\tfrac{1}{4}\lambda_2\sum_{p,q}\theta^p_q\theta^q_p \tag{2}$$

where

$$\theta^p_q=\sum_{i,j}\varphi_{ijq}\varphi^{ijp}. \tag{3}$$

Taking into account the antisymmetry of $\varphi_{ijk}$ the conditions for critical points, namely

$$\partial V/\partial\varphi_{ijk}=0, \tag{4}$$

take the form (Jetzer *et al* 1983)

$$0=\left(-\mu^2+\lambda_1\sum_p \theta_p^p\right)\varphi^{ijk}+\tfrac{1}{3}\lambda_2\sum_p(\theta_p^i\varphi^{pjk}+\theta_p^j\varphi^{ipk}+\theta_p^k\varphi^{ijp}). \tag{5}$$

Contracting with $\varphi_{ijk}$ gives

$$\left(-\mu^2+\lambda_1\sum_p\theta_p^p\right)\sum_p\theta_p^p+\lambda_2\sum_{p,q}\theta_p^q\theta_q^p=0 \tag{6}$$

so that substitution in (2) gives the value of $V(\varphi)$ at the corresponding critical point

$$V(\varphi)=\tfrac{1}{4}\mu^2\sum_p\theta_p^p. \tag{7}$$

Defining $\xi$ (for $\lambda_2\neq 0, \varphi\neq 0$) by

$$\mu^2=(\lambda_1+\xi\lambda_2)\sum_p\theta_p^p \tag{8}$$

and substituting in (6) gives (cf Kim 1982b)

$$\sum_{p,q}\theta_p^q\theta_q^p=\xi\left(\sum_p\theta_p^p\right)^2. \tag{9}$$

Next the Hermitian matrix $\theta_q^p$ may be diagonalised by a suitable SU($n$) transformation (which preserves the form (3) of the relationship between $\theta_q^p$ and $\varphi_{ijk}$), so that (no summation over $p$)

$$\theta_q^p=\theta_p^p\delta_q^p \tag{10}$$

with $\theta_p^p\geqslant 0$. Using this diagonal form of $\theta_q^p$, (9) becomes

$$\sum_p(\theta_p^p)^2=\xi\left(\sum_p\theta_p^p\right)^2 \tag{11}$$

and the critical point conditions (5) are just

$$\left(-\mu^2+\lambda_1\sum_p\theta_p^p+\tfrac{1}{3}\lambda_2(\theta_r^r+\theta_s^s+\theta_t^t)\right)\varphi^{rst}=0 \tag{12}$$

with no summation over $r, s, t$. It follows, using (8), that for all triples $(r, s, t)$ such that $\phi_{rst}\neq 0$:

$$\tfrac{1}{3}(\theta_r^r+\theta_s^s+\theta_t^t)=\xi\sum_p\theta_p^p. \tag{13}$$

From (11) and (13), using the fact that in (10), $\theta_p^p\geqslant 0$ for all $p$, we obtain the bounds

$$1/n\leqslant\xi\leqslant\tfrac{1}{3}. \tag{14}$$

Using (7) and (8) gives the value of $V(\varphi)$ in terms of the parameter $\xi$ at the corresponding critical point

$$V(\varphi)=-\tfrac{1}{4}\mu^4/(\lambda_1+\xi\lambda_2). \tag{15}$$

The conditions for the potential to be bounded below and the limits on $\xi$ mean that the asymptote of $V(\varphi)$ always occurs outside the range of $\xi$. Thus it is found that $V(\varphi)$ decreases or increases monotonically with $\xi$ depending on the sign of $\lambda_2$ and so we get the following bounds:

$$V(\varphi) \geqslant -\tfrac{1}{4}\mu^4/(\lambda_1 + \tfrac{1}{3}\lambda_2) \tag{16}$$

for $-3\lambda_1 < \lambda_2 < 0$ and

$$V(\varphi) \geqslant -\tfrac{1}{4}\mu^4/[\lambda_1 + (1/n)\lambda_2] \tag{17}$$

for $\lambda_2 > 0$ and $n\lambda_1 + \lambda_2 > 0$.

Note that the first bound can only be attained, for all $n$, by having one component of $\varphi$ non-zero, say $\varphi_{123} = a$.

The canonical form of a second-rank antisymmetric tensor $\varphi_{ij}$ is well known and it is convenient to specify the non-vanishing components of such a form $\varphi_{12} = a$, $\varphi_{34} = b, \ldots$ with a maximum of $[n/2]$ such terms. It appears that no such form is available in the literature for the case of a third-rank antisymmetric tensor $\varphi_{ijk}$. The form $\varphi_{123} = a$, $\varphi_{456} = b, \ldots$ (Kim 1982a) is not canonical, but the conjecture by Jetzer *et al* (1983) that the required form is always such that $\varphi_{ijk} \neq 0$ and $\varphi_{ijm} \neq 0$ implies that $k = m$ may well be true. This guarantees that $\theta^p_q$ is diagonal, as required by the above analysis, and enables many candidate solutions to be studied. Moreover this conjecture is consistent with the canonical forms which are available for GL($n$) for specific values of $n$. Following Schouten (1931) and Gurevich (1934, 1935, 1964) we consider the 23 forms for $4 \leqslant n \leqslant 8$. In each case there is a set of values of the non-vanishing components of the form which corresponds to the critical points of (1). Equation (13) fixes the solution up to a constant $a$, which may be found using (8), and the corresponding value of $\xi$ is uniquely determined.

For $n = 4$ there is only one non-vanishing solution $\varphi_{123} = a$, leading to $\xi = \frac{1}{3}$.

For $n = 5$ there are two solutions $\varphi_{123} = a$ and $\varphi_{123} = \varphi_{145} = a$ leading to $\xi = \frac{1}{3}$ and $\xi = \frac{2}{9}$ respectively.

For $n = 6$ there are four solutions $\varphi_{123} = a$, $\varphi_{123} = \varphi_{145} = a$, $\varphi_{123} = \varphi_{145} = \varphi_{246} = a$, and $\varphi_{123} = \varphi_{456} = a$ giving $\xi = \frac{1}{3}, \frac{2}{9}, \frac{5}{27}$ and $\frac{1}{6}$ respectively. The latter corresponds to the bound (17) on the absolute minimum in the case $\lambda_2 > 0$.

For $n = 7$ there are nine solutions with $\xi$ ranging from $\frac{1}{3}$ to $\frac{1}{7}$, and for $n = 8$, 22 solutions with $\xi$ ranging from $\frac{1}{3}$ to $\frac{1}{8}$. The bound (17) on the absolute minimum for $\lambda_2 > 0$ is attained in these two cases through the solutions

$$\varphi_{123} = \varphi_{145} = \varphi_{167} = a, \qquad \varphi_{246} = \varphi_{357} = \sqrt{2}a,$$

for $n = 7$ and

$$\varphi_{123} = \varphi_{145} = \sqrt{3}a, \qquad \varphi_{246} = \varphi_{357} = \sqrt{2}a, \qquad \varphi_{678} = 2a, \qquad \varphi_{258} = \varphi_{348} = a,$$

for $n = 8$.

The identity component of the little groups of the various critical points found as solutions to (17) may be determined by noting that the generators of the little group take the form $\sum_{i,j} \gamma^i_j E^j_i$, where $E^j_i$ for $i, j = 1, 2, \ldots, n$ are the standard generators of GL($n$) and the constraint $\gamma^i_j = \gamma^{j*}_i$ ensures that they belong to U($n$). The coefficients

$\gamma_j^i$ which characterise the algebra of the little group $H$ are those for which

$$\sum_p (\gamma_i^p \varphi_{pjk} + \gamma_j^p \varphi_{ipk} + \gamma_k^p \varphi_{ijp}) = 0 \tag{18}$$

for all $i, j, k$.

Statements here about symmetry breaking patterns should be viewed as statements about the identity component of the relevant little groups.

This is in line with the convention adopted in most gauge theoretic calculations of symmetry breaking. For $n \geq 4$ and $-3\lambda_1 < \lambda_2 < 0$ the little group corresponding to the absolute minimum is $SU(3) \times U(n-3)$ if $G$ is $U(n)$ and $SU(3) \times SU(n-3)$ when $G$ is $SU(n)$. It only remains to consider the case $\lambda_2 > 0$. This restriction is assumed henceforth.

Apart from the distinction between $U(n)$ and $SU(n)$ the cases $n = 4, 5$ and 6 have been covered by Jetzer *et al* (1983). The absolute minima correspond to symmetry breaking from U(4) to $SU(3) \times U(1)$ and SU(4) to SU(3), from U(5) to $Sp(4) \times U(1)$ and from SU(5) to Sp(4), and from both U(6) and SU(6) to $SU(3) \times SU(3)$. In the case of U(6) the little group is that of $\varphi_{123} = \varphi_{456} = a$ and illustrates the difference between a maximal stationary subalgebra and a maximal little group. The Lie algebra of this little group is strictly contained in the Lie algebra of another little group $SU(3) \times U(3)$, corresponding to $\varphi_{123} = a$. However, there is a discrete group $Z_2$ generated by $(14)\,(25)\,(36) \in U(6)$ which leaves the first point fixed, but not the second. So the little group of $\varphi_{123} = \varphi_{456} = a$ is not contained in the little group of $\varphi_{123} = a$. In fact both are maximal so that Michel's conjecture is not violated. However, the analogue of the conjecture in the case of Lie algebras is clearly violated by this U(6) example and the conjecture loses much of its usefulness.

In the case of U(7) and SU(7) the various little groups of the critical points, together with the corresponding values of $\xi$, are shown in figure 1. The absolute minima involves symmetry breaking from U(7) and SU(7) to $G_2$, which is a maximal little group.

The corresponding results for U(8) and SU(8) are very complicated and will not be presented here. It suffices to say that the little group of the absolute minimum is the maximal little group SU(3) embedded in both U(8) and SU(8) via its adjoint representation.

For $n = 9$ and $\lambda_2 > 0$ it transpires that one solution attaining the bound (17) is easy to write down. The solution is $\varphi_{123} = \varphi_{456} = \varphi_{789} = a$, first given by Kim (1982a), which leads to breaking from U(9) and SU(9) to $SU(3) \times SU(3) \times SU(3)$. However, there are at least three others attaining the same bound corresponding to breaking to $SU(3) \times U(1) \times U(1)$, or to SU(3), or to $U(1) \times U(1) \times U(1) \times U(1)$. Of the corresponding stationary subalgebras the last two are not maximal although the first is maximal. Thus for example the solution $\varphi_{123} = \varphi_{145} = \varphi_{167} = \varphi_{189} = \varphi_{249} = \varphi_{256} = \varphi_{278} = \varphi_{347} = \varphi_{369} = \varphi_{358} = \varphi_{468} = \varphi_{579} = a$ provides an example of an absolute minimum with breaking to $U(1) \times U(1) \times U(1) \times U(1)$ whose corresponding stationary subalgebra is not maximal.

More surprisingly the case $n = 10$ yields an absolute minimum whose little group can only be finite since all the coefficients $\gamma_j^i$ in (18) are zero. This solution is $\varphi_{123} = \varphi_{4610} = \varphi_{5910} = \varphi_{7810} = \sqrt{3}a$, $\varphi_{145} = \varphi_{167} = \varphi_{189} = \varphi_{247} = \varphi_{269} = \varphi_{258} = \varphi_{349} = \varphi_{368} = \varphi_{357} = a$.

The bound (17) on $V(\varphi)$ is attained for all $n > 6$ for at least one solution to the critical point conditions (4). This can be seen by combining the solutions for $n = 7, 8, 9$ with $\varphi_{n-2,\,n-1,\,n} = \varphi_{n-5,\,n-4,\,n-3} = \ldots \varphi_{n-3k-2,\,n-3k-1,\,n-3k} = a$ for $k = [n/3] - 2$. In general other, degenerate, solutions exist. We note that for the cases $n \geq 9$ the degenerate

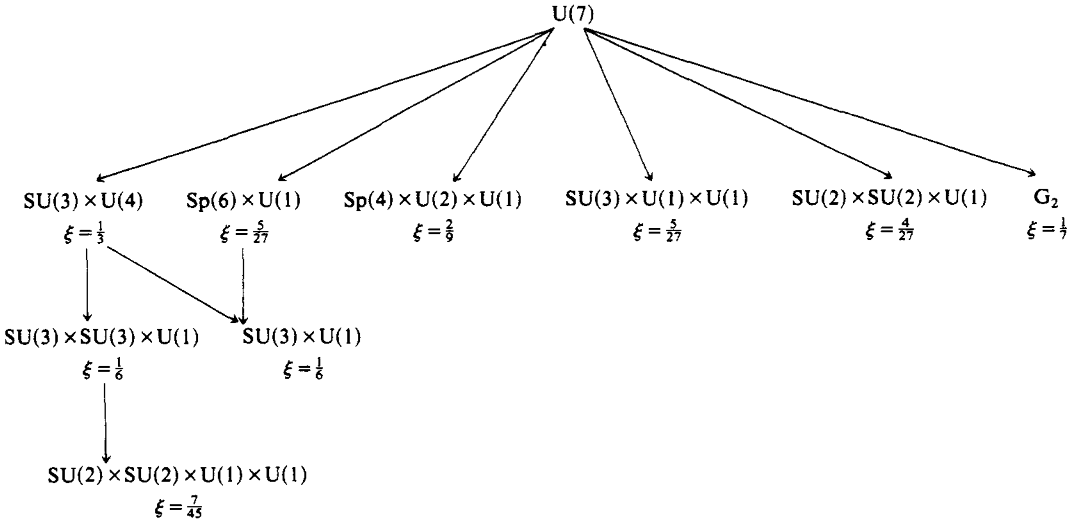


**Figure 1.** (*a*) Identity components of little groups of critical point solutions derived from Gurevich's canonical forms as subgroups of U(7).

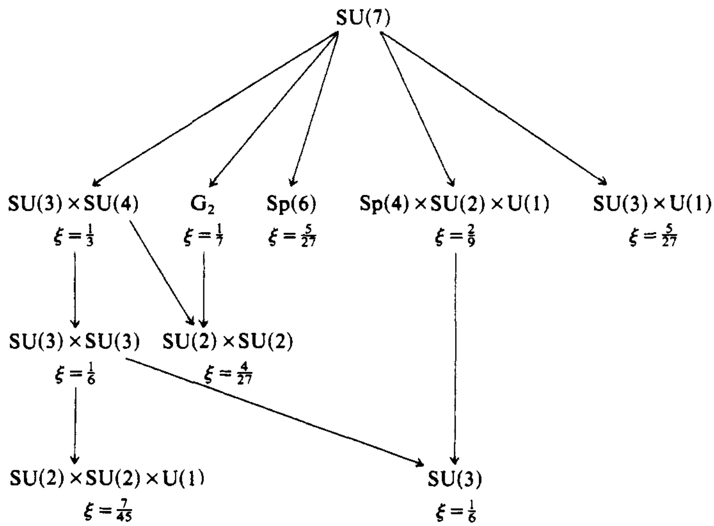


**Figure 1.** (*b*) Identity components of little groups of critical point solutions derived from Gurevich's canonical forms as subgroups of SU(7).

minima probably give rise to pseudo-Goldstone bosons (Georgi and Pais 1975) and that an effective potential calculation would be necessary to determine the true symmetry breaking pattern.

Although counter-examples to the conjecture of Michel exist in the case of discrete groups $G$ (Jaric 1983), we have been unable to produce a clearcut counter-example in the case of compact continuous Lie groups. We speculate that a more complete analysis using canonical forms for $n \geqslant 9$ and dealing with finite subgroups could produce such an example.

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